

Exact Dynamical Structure Factor of a Bose Liquid

Girish S. Setlur

Department of Physics and Materials Research Laboratory

University of Illinois at Urbana-Champaign, Urbana, Illinois 61801

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Based on ideas introduced in a previous preprint cond-mat/9701206 we propose an exactly solvable model of bosons interacting amongst themselves via a Van-der Waal-like repulsive interaction, and compute both the filling fraction and the dynamical structure factor exactly. The novelty of this approach involves introducing, analogous to Fermi sea (or surface) displacements, Bose fields that in this case, correspond to fluctuations of the Bose condensate. The exact dynamical structure factor has a coherent part that corresponds to the Bogoliubov spectrum and an incoherent part that is a result of correlations.

I. INTRODUCTION

The microscopic theory of He4 is by now a well established and extremely succesful theory. This theory is the culmination of the efforts of a large number of distinguished physicists starting from Bogoliubov¹ whose theory of a weakly non-ideal bose gas was the inspiration for all the theoretical work that followed in this field. However, the modern theory of superfluidity as it is understood today was poineered by Penrose and Onsager² who introduced the concept of off-diagonal long range order. First attempts at guessing the form of the excitation spectrum was made by Bijl and Feynman³. Landau⁴ surmised many of the properties of the excitation spectrum including the phonon and the roton parts before it was measured experimaentally. The microscopic theory for obtaining the static structure factor employs correlated basis functions that was pioneered by Bijl, Dingje and Jastrow⁵. An elaborate application of this technique to quantum liquids in general may be found in the book by Feenberg⁶. Also the work of Kebukawa, Sunakawa and Yamasaki⁷ in computing the dynamical properties including the excitation spectrum, backflow e.t.c. has to be mentioned.

In the light of these historical remarks, it seems that yet another theory of this system is redundant. However, there is always room for a fresh perspective on well understood problems. The hope is that the approach that we are now going to describe along with its fermionic counterpart (described elsewhere⁸) will produce a model of matter that has these desirable qualities, namely, it is exactly solvable, captures the important physics and contains no phenomenological parameters other than the electric charge and mass. If thse types of models were to prove succesful, an outcome that is tantalisingly plausible, it will be a vindication of the soundness of the reductionist philosophy.

II. BOSONS AT ZERO TEMPERATURE

In this article we do what we did earlier (cond-mat/9701206) except that here we are bosonizing the bosons. Let $b_{\mathbf{k}}$ and $b_{\mathbf{k}}^\dagger$ be the Bose fields in question. Analogous to Fermi sea displacements we introduce Bose condensate displacements ($d_{\mathbf{k}}(\mathbf{q})$) which are also bose fields obeying canonical commutation relations identical to its parents $b_{\mathbf{k}}$ and $b_{\mathbf{k}}^\dagger$.

$$\begin{aligned}
b_{\mathbf{k}+\mathbf{q}/2}^\dagger b_{\mathbf{k}-\mathbf{q}/2} &= N \delta_{\mathbf{k},0} \delta_{\mathbf{q},0} + (\sqrt{N}) [\delta_{\mathbf{k}+\mathbf{q}/2,0} d_{\mathbf{k}}(-\mathbf{q}) + \delta_{\mathbf{k}-\mathbf{q}/2,0} d_{\mathbf{k}}^\dagger(\mathbf{q})] \\
&+ \sum_{\mathbf{q}_1} T_1(\mathbf{k}, \mathbf{q}, \mathbf{q}_1) d_{\mathbf{k}+\mathbf{q}/2-\mathbf{q}_1/2}^\dagger(\mathbf{q}_1) d_{\mathbf{k}-\mathbf{q}_1/2}(-\mathbf{q} + \mathbf{q}_1) \\
&+ \sum_{\mathbf{q}_1} T_2(\mathbf{k}, \mathbf{q}, \mathbf{q}_1) d_{\mathbf{k}-\mathbf{q}/2+\mathbf{q}_1/2}^\dagger(\mathbf{q}_1) d_{\mathbf{k}+\mathbf{q}_1/2}(-\mathbf{q} + \mathbf{q}_1)
\end{aligned} \tag{1}$$

The above relation is meant to be an operator identity. In other words, all dynamical moments of the product $b_{\mathbf{k}+\mathbf{q}/2}^\dagger b_{\mathbf{k}-\mathbf{q}/2}$ are equal when evaluated either in the original bose language or in the condensate-displacement language. This is true provided we identify the vacuum of the $d_{\mathbf{k}}(\mathbf{q})$ with the bose condensate (the ground state of the noninteracting system).

$$d_{\mathbf{k}}(\mathbf{q})|BC\rangle = 0 \tag{2}$$

Furthermore, the chemical potential of the condensate-displacement bosons is taken to be zero. In other words the number of condensate displacement bosons is not conserved.

To make all the dynamical moments of $b_{\mathbf{k}+\mathbf{q}/2}^\dagger b_{\mathbf{k}-\mathbf{q}/2}$ come out right and the commutation rules amongst them also come out right provided we choose T_1 and T_2 such that,

$$\begin{aligned}
&(\sqrt{N})^2 \delta_{\mathbf{k}+\mathbf{q}/2,0} \delta_{\mathbf{k}''-\mathbf{q}''/2,0} \sum_{\mathbf{q}_1} T_1(\mathbf{k}', \mathbf{q}', \mathbf{q}_1) \langle d_{\mathbf{k}}(-\mathbf{q}) d_{\mathbf{k}'+\mathbf{q}'/2-\mathbf{q}_1/2}^\dagger(\mathbf{q}_1) d_{\mathbf{k}'-\mathbf{q}_1/2}(-\mathbf{q}' + \mathbf{q}_1) d_{\mathbf{k}''}^\dagger(\mathbf{q}'') \rangle \\
&+ (\sqrt{N})^2 \delta_{\mathbf{k}+\mathbf{q}/2,0} \delta_{\mathbf{k}''-\mathbf{q}''/2,0} \sum_{\mathbf{q}_1} T_2(\mathbf{k}', \mathbf{q}', \mathbf{q}_1) \langle d_{\mathbf{k}}(-\mathbf{q}) d_{\mathbf{k}'-\mathbf{q}'/2+\mathbf{q}_1/2}^\dagger(\mathbf{q}_1) d_{\mathbf{k}'+\mathbf{q}_1/2}(-\mathbf{q}' + \mathbf{q}_1) d_{\mathbf{k}''}^\dagger(\mathbf{q}'') \rangle \\
&= \\
&N \delta_{\mathbf{k}+\mathbf{q}/2,0} \delta_{\mathbf{k}+\mathbf{q}/2,\mathbf{k}''-\mathbf{q}''/2} \delta_{\mathbf{k}-\mathbf{q}/2,\mathbf{k}'+\mathbf{q}'/2} \delta_{\mathbf{k}'-\mathbf{q}'/2,\mathbf{k}''+\mathbf{q}''/2}
\end{aligned} \tag{3}$$

This means,

$$T_1(-\mathbf{q} - \mathbf{q}'/2, \mathbf{q}', -\mathbf{q}) = 1; \mathbf{q} \neq 0; \mathbf{q}' \neq 0 \tag{4}$$

$$T_2(-\mathbf{q}'/2, \mathbf{q}', -\mathbf{q}) = 0; \mathbf{q} \neq 0; \mathbf{q}' \neq 0 \tag{5}$$

in order for the kinetic energy operator to have the form,

$$K = \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} d_{(1/2)\mathbf{k}}^{\dagger}(\mathbf{k}) d_{(1/2)\mathbf{k}}(\mathbf{k}) \quad (6)$$

we must have,

$$\epsilon_{\mathbf{k}+\mathbf{q}_1/2} T_1(\mathbf{k} + \mathbf{q}_1/2, \mathbf{0}, \mathbf{q}_1) + \epsilon_{\mathbf{k}-\mathbf{q}_1/2} T_2(\mathbf{k} - \mathbf{q}_1/2, \mathbf{0}, \mathbf{q}_1) = \delta_{\mathbf{k}, \mathbf{q}_1/2} \epsilon_{\mathbf{q}_1} \quad (7)$$

In order for,

$$\sum_{\mathbf{k}} b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} = N$$

we must have,

$$T_1(\mathbf{k} + \mathbf{q}_1/2, \mathbf{0}, \mathbf{q}_1) + T_2(\mathbf{k} - \mathbf{q}_1/2, \mathbf{0}, \mathbf{q}_1) = 0 \quad (8)$$

In order for the commutation rules amongst the $b_{\mathbf{k}+\mathbf{q}/2}^{\dagger} b_{\mathbf{k}-\mathbf{q}/2}$ to come out right, we must have in addition to all the above relations,

$$T_2(\mathbf{q}/2, \mathbf{q}, -\mathbf{q}') = 0 \quad (9)$$

$$T_2(-\mathbf{q}/2, \mathbf{q}, \mathbf{q}') = 0 \quad (10)$$

Any choice of T_1 and T_2 consistent with the above relations should suffice. From the above relations (Eq.(7) and Eq.(8)), we have quite unambiguously,

$$T_1(\mathbf{k} + \mathbf{q}_1/2, \mathbf{0}, \mathbf{q}_1) = \delta_{\mathbf{k}, \mathbf{q}_1/2} \quad (11)$$

$$T_2(\mathbf{k} - \mathbf{q}_1/2, \mathbf{0}, \mathbf{q}_1) = -\delta_{\mathbf{k}, \mathbf{q}_1/2} \quad (12)$$

$$T_1(-\mathbf{q} - \mathbf{q}'/2, \mathbf{q}', -\mathbf{q}) = 1 \quad (13)$$

and,

$$T_1(\mathbf{k}, \mathbf{q}, \mathbf{q}_1) = 0; \text{ otherwise} \quad (14)$$

$$T_2(\mathbf{k}, \mathbf{q}, \mathbf{q}_1) = 0; \text{ otherwise} \quad (15)$$

This means that we may rewrite the formula for $b_{\mathbf{k}+\mathbf{q}/2}^{\dagger} b_{\mathbf{k}-\mathbf{q}/2}$ as follows,

$$\begin{aligned} b_{\mathbf{k}+\mathbf{q}/2}^{\dagger} b_{\mathbf{k}-\mathbf{q}/2} &= N \delta_{\mathbf{k},0} \delta_{\mathbf{q},0} + (\sqrt{N})(1 - \delta_{\mathbf{k},0} \delta_{\mathbf{q},0}) [\delta_{\mathbf{k}+\mathbf{q}/2,0} d_{\mathbf{k}}(-\mathbf{q}) + \delta_{\mathbf{k}-\mathbf{q}/2,0} d_{\mathbf{k}}^{\dagger}(\mathbf{q})] \\ &\quad + d_{(1/2)\mathbf{k}+\mathbf{q}/2}^{\dagger}(\mathbf{k} + \mathbf{q}/2) d_{(1/2)\mathbf{k}-\mathbf{q}/2}(\mathbf{k} - \mathbf{q}/2) \end{aligned}$$

$$- \delta_{\mathbf{k},0} \delta_{\mathbf{q},0} \sum_{\mathbf{q}_1} d_{\mathbf{q}_1/2}^\dagger(\mathbf{q}_1) d_{\mathbf{q}_1/2}(\mathbf{q}_1) \quad (16)$$

Consider an interaction of the type,

$$H_I = \left(\frac{\rho_0}{2}\right) \sum_{\mathbf{q} \neq 0} v_{\mathbf{q}} \sum_{\mathbf{k}, \mathbf{k}'} [\delta_{\mathbf{k}+\mathbf{q}/2,0} d_{\mathbf{k}}(-\mathbf{q}) + \delta_{\mathbf{k}-\mathbf{q}/2,0} d_{\mathbf{k}}^\dagger(\mathbf{q})] [\delta_{\mathbf{k}'-\mathbf{q}/2,0} d_{\mathbf{k}'}(\mathbf{q}) + \delta_{\mathbf{k}'+\mathbf{q}/2,0} d_{\mathbf{k}'}^\dagger(-\mathbf{q})] \quad (17)$$

or,

$$H_I = \left(\frac{\rho_0}{2}\right) \sum_{\mathbf{q} \neq 0} v_{\mathbf{q}} [d_{-\mathbf{q}/2}(-\mathbf{q}) + d_{\mathbf{q}/2}^\dagger(\mathbf{q})] [d_{\mathbf{q}/2}(\mathbf{q}) + d_{-\mathbf{q}/2}^\dagger(-\mathbf{q})] \quad (18)$$

and the free case is given by,

$$H_0 = \sum_{\mathbf{q}} \epsilon_{\mathbf{q}} d_{(1/2)\mathbf{q}}^\dagger(\mathbf{q}) d_{(1/2)\mathbf{q}}(\mathbf{q}) \quad (19)$$

The full hamiltonian may be diagonalised as follows,

$$H = \sum_{\mathbf{q}} \omega_{\mathbf{q}} f_{\mathbf{q}}^\dagger f_{\mathbf{q}} \quad (20)$$

and,

$$f_{\mathbf{q}} = \left(\frac{\omega_{\mathbf{q}} + \epsilon_{\mathbf{q}} + \rho_0 v_{\mathbf{q}}}{2 \omega_{\mathbf{q}}}\right)^{\frac{1}{2}} d_{\mathbf{q}/2}(\mathbf{q}) + \left(\frac{-\omega_{\mathbf{q}} + \epsilon_{\mathbf{q}} + \rho_0 v_{\mathbf{q}}}{2 \omega_{\mathbf{q}}}\right)^{\frac{1}{2}} d_{-\mathbf{q}/2}^\dagger(-\mathbf{q}) \quad (21)$$

$$f_{-\mathbf{q}}^\dagger = \left(\frac{-\omega_{\mathbf{q}} + \epsilon_{\mathbf{q}} + \rho_0 v_{\mathbf{q}}}{2 \omega_{\mathbf{q}}}\right)^{\frac{1}{2}} d_{\mathbf{q}/2}(\mathbf{q}) + \left(\frac{\omega_{\mathbf{q}} + \epsilon_{\mathbf{q}} + \rho_0 v_{\mathbf{q}}}{2 \omega_{\mathbf{q}}}\right)^{\frac{1}{2}} d_{-\mathbf{q}/2}^\dagger(-\mathbf{q}) \quad (22)$$

$$d_{\mathbf{q}/2}(\mathbf{q}) = \left(\frac{\omega_{\mathbf{q}} + \epsilon_{\mathbf{q}} + \rho_0 v_{\mathbf{q}}}{2 \omega_{\mathbf{q}}}\right)^{\frac{1}{2}} f_{\mathbf{q}} - \left(\frac{-\omega_{\mathbf{q}} + \epsilon_{\mathbf{q}} + \rho_0 v_{\mathbf{q}}}{2 \omega_{\mathbf{q}}}\right)^{\frac{1}{2}} f_{-\mathbf{q}}^\dagger \quad (23)$$

$$d_{-\mathbf{q}/2}^\dagger(-\mathbf{q}) = \left(\frac{\omega_{\mathbf{q}} + \epsilon_{\mathbf{q}} + \rho_0 v_{\mathbf{q}}}{2 \omega_{\mathbf{q}}}\right)^{\frac{1}{2}} f_{-\mathbf{q}}^\dagger - \left(\frac{-\omega_{\mathbf{q}} + \epsilon_{\mathbf{q}} + \rho_0 v_{\mathbf{q}}}{2 \omega_{\mathbf{q}}}\right)^{\frac{1}{2}} f_{\mathbf{q}} \quad (24)$$

$$\omega_{\mathbf{q}} = \sqrt{\epsilon_{\mathbf{q}}^2 + 2\rho_0 v_{\mathbf{q}} \epsilon_{\mathbf{q}}} \quad (25)$$

From this we may deduce,

$$\langle d_{(1/2)\mathbf{q}}^\dagger(\mathbf{q}) d_{(1/2)\mathbf{q}}(\mathbf{q}) \rangle = \frac{-\omega_{\mathbf{q}} + \epsilon_{\mathbf{q}} + \rho_0 v_{\mathbf{q}}}{2 \omega_{\mathbf{q}}} \quad (26)$$

It is worth while to note that $\epsilon_{\mathbf{q}} + \rho_0 v_{\mathbf{q}} > \omega_{\mathbf{q}}$. From this it is possible to write down the filling fraction.

FILLING FRACTION

$$f_0 = N_0/N = 1 - (1/N) \sum_{\mathbf{q}} \langle d_{(1/2)\mathbf{q}}^\dagger(\mathbf{q}) d_{(1/2)\mathbf{q}}(\mathbf{q}) \rangle \quad (27)$$

or,

$$f_0 = N_0/N = 1 - (1/2\pi^2\rho_0) \int_0^\infty dq \, q^2 \left(\frac{-\omega_{\mathbf{q}} + \epsilon_{\mathbf{q}} + \rho_0 v_{\mathbf{q}}}{2 \omega_{\mathbf{q}}} \right) \quad (28)$$

First define,

$$A_{\mathbf{q}} = \left(\frac{\omega_{\mathbf{q}} + \epsilon_{\mathbf{q}} + \rho_0 v_{\mathbf{q}}}{2 \omega_{\mathbf{q}}} \right)^{\frac{1}{2}} \quad (29)$$

$$B_{\mathbf{q}} = \left(\frac{-\omega_{\mathbf{q}} + \epsilon_{\mathbf{q}} + \rho_0 v_{\mathbf{q}}}{2 \omega_{\mathbf{q}}} \right)^{\frac{1}{2}} \quad (30)$$

The density operator is,

$$\rho_{\mathbf{q}}(t) = \sqrt{N} [d_{-(1/2)\mathbf{q}}(-\mathbf{q})(t) + d_{(1/2)\mathbf{q}}^{\dagger}(\mathbf{q})(t)] + \sum_{\mathbf{k}} d_{(1/2)\mathbf{k}+\mathbf{q}/2}^{\dagger}(\mathbf{k} + \mathbf{q}/2)(t) d_{(1/2)\mathbf{k}-\mathbf{q}/2}(\mathbf{k} - \mathbf{q}/2)(t) \quad (31)$$

$$d_{-(1/2)\mathbf{q}}(-\mathbf{q})(t) = A_{\mathbf{q}} f_{-\mathbf{q}} e^{-i \omega_{\mathbf{q}} t} - B_{\mathbf{q}} f_{\mathbf{q}}^{\dagger} e^{i \omega_{\mathbf{q}} t} \quad (32)$$

$$d_{(1/2)\mathbf{q}}^{\dagger}(\mathbf{q})(t) = A_{\mathbf{q}} f_{\mathbf{q}}^{\dagger} e^{i \omega_{\mathbf{q}} t} - B_{\mathbf{q}} f_{-\mathbf{q}} e^{-i \omega_{\mathbf{q}} t} \quad (33)$$

$$d_{(1/2)\mathbf{k}-\mathbf{q}/2}(\mathbf{k} - \mathbf{q}/2)(t) = A_{\mathbf{k}-\mathbf{q}/2} f_{\mathbf{k}-\mathbf{q}/2} e^{-i \omega_{\mathbf{k}-\mathbf{q}/2} t} - B_{\mathbf{k}-\mathbf{q}/2} f_{-\mathbf{k}+\mathbf{q}/2}^{\dagger} e^{i \omega_{\mathbf{k}-\mathbf{q}/2} t} \quad (34)$$

$$d_{(1/2)\mathbf{k}+\mathbf{q}/2}^{\dagger}(\mathbf{k} + \mathbf{q}/2)(t) = A_{\mathbf{k}+\mathbf{q}/2} f_{\mathbf{k}+\mathbf{q}/2}^{\dagger} e^{i \omega_{\mathbf{k}+\mathbf{q}/2} t} - B_{\mathbf{k}+\mathbf{q}/2} f_{-\mathbf{k}-\mathbf{q}/2} e^{-i \omega_{\mathbf{k}+\mathbf{q}/2} t} \quad (35)$$

Now define,

$$S^>(\mathbf{q}t) = \langle \rho_{\mathbf{q}}(t) \rho_{-\mathbf{q}}(0) \rangle = N \langle [d_{-(1/2)\mathbf{q}}(-\mathbf{q})(t) + d_{(1/2)\mathbf{q}}^{\dagger}(\mathbf{q})(t)] [d_{(1/2)\mathbf{q}}(\mathbf{q})(0) + d_{-(1/2)\mathbf{q}}^{\dagger}(-\mathbf{q})(0)] \rangle$$

$$\sum_{\mathbf{k}, \mathbf{k}'} \langle d_{(1/2)\mathbf{k}+\mathbf{q}/2}^{\dagger}(\mathbf{k} + \mathbf{q}/2)(t) d_{(1/2)\mathbf{k}-\mathbf{q}/2}(\mathbf{k} - \mathbf{q}/2)(t) d_{(1/2)\mathbf{k}'-\mathbf{q}/2}^{\dagger}(\mathbf{k}' - \mathbf{q}/2)(0) d_{(1/2)\mathbf{k}'+\mathbf{q}/2}(\mathbf{k}' + \mathbf{q}/2)(0) \rangle$$

$$= N \left(\frac{\epsilon_{\mathbf{q}}}{\omega_{\mathbf{q}}} \right) \exp(-i \omega_{\mathbf{q}} t)$$

$$+ \sum_{\mathbf{k}, \mathbf{k}'} \langle B_{\mathbf{k}+\mathbf{q}/2} f_{-\mathbf{k}-\mathbf{q}/2} e^{-i \omega_{\mathbf{k}+\mathbf{q}/2} t} A_{\mathbf{k}-\mathbf{q}/2} f_{\mathbf{k}-\mathbf{q}/2} e^{-i \omega_{\mathbf{k}-\mathbf{q}/2} t} A_{\mathbf{k}'-\mathbf{q}/2} f_{\mathbf{k}'-\mathbf{q}/2}^{\dagger} B_{\mathbf{k}'+\mathbf{q}/2} f_{-\mathbf{k}'-\mathbf{q}/2}^{\dagger} \rangle$$

$$+ \sum_{\mathbf{k}, \mathbf{k}'} \langle B_{\mathbf{k}+\mathbf{q}/2} f_{-\mathbf{k}-\mathbf{q}/2} e^{-i \omega_{\mathbf{k}+\mathbf{q}/2} t} B_{\mathbf{k}-\mathbf{q}/2} f_{-\mathbf{k}+\mathbf{q}/2}^{\dagger} e^{i \omega_{\mathbf{k}-\mathbf{q}/2} t} B_{\mathbf{k}'-\mathbf{q}/2} f_{-\mathbf{k}'+\mathbf{q}/2} B_{\mathbf{k}'+\mathbf{q}/2} f_{-\mathbf{k}'-\mathbf{q}/2}^{\dagger} \rangle$$

$$S^>(\mathbf{q}, t) = N \left(\frac{\epsilon_{\mathbf{q}}}{\omega_{\mathbf{q}}} \right) \exp(-i \omega_{\mathbf{q}} t)$$

$$+ \sum_{\mathbf{k}} \exp(-i (\omega_{\mathbf{k}+\mathbf{q}/2} + \omega_{\mathbf{k}-\mathbf{q}/2}) t) [B_{\mathbf{k}+\mathbf{q}/2}^2 A_{\mathbf{k}-\mathbf{q}/2}^2 + B_{\mathbf{k}+\mathbf{q}/2} B_{-\mathbf{k}+\mathbf{q}/2} A_{\mathbf{k}-\mathbf{q}/2} A_{\mathbf{k}+\mathbf{q}/2}]$$

$$S^<(\mathbf{q}, t) = N \left(\frac{\epsilon_{\mathbf{q}}}{\omega_{\mathbf{q}}} \right) \exp(i \omega_{\mathbf{q}} t) + \sum_{\mathbf{k}} \exp(i (\omega_{\mathbf{k}+\mathbf{q}/2} + \omega_{\mathbf{k}-\mathbf{q}/2}) t)$$

$$[B_{\mathbf{k}-\mathbf{q}/2}^2 A_{\mathbf{k}+\mathbf{q}/2}^2 + B_{\mathbf{k}-\mathbf{q}/2} B_{-\mathbf{k}-\mathbf{q}/2} A_{\mathbf{k}+\mathbf{q}/2} A_{\mathbf{k}-\mathbf{q}/2}]$$

and,

$$S^>(\mathbf{q}, t) = \langle \rho_{\mathbf{q}}(t) \rho_{-\mathbf{q}}(0) \rangle$$

$$S^<(\mathbf{q}, t) = \langle \rho_{-\mathbf{q}}(0) \rho_{\mathbf{q}}(t) \rangle$$

From the above equations, it is easy to see that there is a coherent part corresponding to the Bogoliubov spectrum and an incoherent part which is due to correlations and is responsible (hopefully) for the roton minimum.

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